

*To Leonid Volevich in occasion of his 70th Birthday*

# Sharp Spectral Asymptotics for Magnetic Schrödinger Operator with Irregular Potential.

Victor Ivrii<sup>\*†</sup>

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## Abstract

In this paper I consider sharp spectral asymptotics for multidimensional magnetic Schrödinger operator with irregular coefficients with respect to two parameters – semi-classical parameter  $h$  and coupling parameter  $\mu$ . There are few principally different cases, depending on dimension, rank of magnetic intensity matrix, relation between  $h$  and  $\mu$  and some extra assumptions.

## 0 Introduction

### 0.1 Preface

In this paper I consider multidimensional Schrödinger operator

$$(0.1) \quad A = A_0 + V(x), \quad A_0 = \sum_{j,k \leq d} P_j g^{jk}(x) P_k,$$

$$P_j = hD_j - \mu V_j(x), \quad h \in (0, 1], \quad \mu \geq 1.$$

It is characterized by *magnetic field intensity matrix*  $(F_{jk})$  with  $F_{jk} = (\partial_{x_j} V_k - \partial_{x_k} V_j)$ , which is skew-symmetric  $d \times d$ -matrix, and  $(F_m^j) = (g^{jk})(F_{km})$  which is unitarily equivalent to

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skew-symmetric matrix  $(g^{jk})^{\frac{1}{2}}(F_{jk})(g^{jk})^{\frac{1}{2}}$ . Then all eigenvalues of  $(F_k^j)$  (with multiplicities) are  $\pm if_m$  ( $f_m > 0$ ,  $m = 1, \dots, r$ ) and 0 of multiplicity  $q = d - 2r$  where  $2r = \text{rank}(F_k^j)$ .

I formulate results only in the case of  $g^{jk} = \delta_{jk}$  (thus covering the case  $g^{jk} = \text{const}$  as well) and  $F_{jk} = \text{const}$  (then one can select linear vector potential  $(V_1(x), \dots, V_d(x))$ ). The results in the general case differ even in their statements. So we consider operator

$$(0.2) \quad \sum_{j,k} P_j^2 + V(x), \quad P_j = hD_j - \mu V_j(x)$$

with linear functions  $V_j(x)$  and constant magnetic intensity matrix  $F_{jk}$ .

The main goal of this paper is to present *the local spectral asymptotics*, i.e. asymptotics of

$$(0.3) \quad \int e(x, x, \tau) \psi(x) dx$$

as  $h \rightarrow +0$ ,  $\mu \rightarrow +\infty$  where  $e(x, y, \tau) = e_{h,\mu}(x, y, \tau)$  is the Schwartz kernel of the spectral projector of  $A$  ( $A$  is assumed to be a self-adjoint operator) and  $\psi$  is a smooth cut-off function supported in the ball  $B(0, \frac{1}{2})$  (all the conjectures save self-adjointness are made for  $B(0, 1)$ ). Combined with partition-rescaling arguments such asymptotics imply many asymptotics of eigenvalue counting function given by formula (0.3) with  $\psi = 1$ .

Under above assumptions in the appropriate coordinates after gauge transform

$$(0.4) \quad V_j(x) = \begin{cases} -f_{j-r}x_{j-r} & \text{as } j = r+1, \dots, 2r \\ 0 & \text{otherwise} \end{cases}$$

where  $f_j > 0$  and  $\pm if_j$  are eigenvalues of  $F_k^j$ ,  $\text{rank } F = 2r$ . Let  $q = d - 2r$ .

In the special case  $V = \text{const}$  everything becomes explicit: after  $h$ -Fourier transform with respect to  $(x_{r+1}, \dots, x_d)$  and change of variables  $x_j \mapsto x_j + \mu^{-1}f_j^{-1}\xi_{2j+1}$  ( $j = 1, \dots, r$ ) operator  $A$  is transformed into

$$(0.5) \quad \sum_{1 \leq j \leq r} (h^2 D_j^2 + \mu^2 f_j^2 x_j^2) + \sum_{1 \leq k \leq q} \xi_{2r+k}^2 + V.$$

Using Hermite function decomposition one can calculate easily  $e(x, y, \tau)$  and prove that

$$(0.6) \quad e(x, x, \tau) = \mathcal{E}_{d,r}^{\text{MW}}(\tau) \stackrel{\text{def}}{=} \omega_q (2\pi)^{-q} \sum_{\alpha \in \mathbf{Z}^{+r}} \left( \tau - \sum_j (2\alpha_j + 1) f_j \mu h - V \right)_+^{\frac{q}{2}} \times \mu^r h^{-d+r} f_1 \cdots f_r \sqrt{g}$$

where  $\omega_q$  is a volume of the unit ball in  $\mathbb{R}^q$ ,  $g = \det(g^{jk})^{-1}$ .

In particular, for  $q = 0$  such operator has pure point spectrum of infinite multiplicity, consisting of *Landau levels*  $E_\alpha = \sum_j (2\alpha_j + 1) f_j \mu h + V$  with  $\alpha \in \mathbb{Z}^{+r}$  while for  $q \geq 1$  the spectrum is absolutely continuous and Landau levels are merely bottoms of its channels  $[E_\alpha, \infty)$ .

Also, as  $\mu h \gg 1$  one needs to include  $-\sum_j f_j \mu h$  or even larger negative number in  $V$  to avoid being in the classically forbidden zone.

Our goal is to study operator  $A$  with variable potential  $V(x)$ ; however the deep distinctions between cases  $\text{rank } F_{jk} = d$  and  $\text{rank } F_{jk} < d$  and between cases  $\mu h \ll 1$ ,  $\mu h \gg 1$  (and intermediate case  $\mu h \sim 1$ ) are preserved as well as the importance of Landau levels. Also,  $\mathcal{E}^{\text{MW}}$  gives a good approximation to  $e(x, x, \tau)$  even if in some cases we need some corrections to derive sharper remainder estimate.

## 0.2 Classical Dynamics

Considering classical trajectories on some finite energy level one can prove easily that for  $V = \text{const}$

(i) As  $\text{rank } F = d = 2$  particles move along circles of the radii  $\asymp \mu^{-1}$ . As  $\text{rank } F = d = 2r$ ,  $r \geq 2$  trajectories are more complicated (generic trajectories are periodic if  $f_1, \dots, f_r$  are commensurable or envelope tori otherwise) but they are confined to  $C\mu^{-1}$ -vicinities of their origins as well;

(ii) As  $d = 3$ ,  $\text{rank } F = 2$  particles move along spirals. Similar description holds for  $\text{rank } F = 2r < d$ ,  $r \geq 1$ : there is a cyclotronic movement as in (i) along  $(x_1, \dots, x_{2r})$  and also a free movement along  $x''' = (x_{2r+1}, \dots, x_d)$ .

So, we see a big difference between full- and not-full-rank cases and more subtle difference between  $d = 2, 3$  and  $d \geq 4$ .

As  $V$  is variable

(iii) As  $\text{rank } F = d = 2$  particles move approximately along circles of the radii  $\asymp \mu^{-1}$  but the centers of these circles are drifting with the speed  $\mu^{-1}|f_1|^{-1}\nabla V^\perp$  (thus the drift is orthogonal to the direction of the electric field). Similarly, as  $\text{rank } F = d = 2r$  the trajectories described in (i) are drifting with the velocity  $\mu^{-1}F^{-1}\nabla V$  where  $F = (F_{jk})$  is the intensity matrix;

(iv) As  $d = 3$ ,  $\text{rank } F = 2$  there is a fast cyclotronic movement along  $(x_1, x_2)$ , temperate movement described by 1-dimensional Hamiltonian  $\xi_3^2 + V(x', x_3) + E_{\text{magn}}$  along  $x_3$  where  $E_{\text{magn}}$  is an energy of the fast movement (which is constant in our assumptions) and a slow drift along all variables. Similar description holds for  $\text{rank } F = 2r < d$ ,  $r \geq 1$ .

So, if  $\text{rank } F = d$  we can follow trajectories until time  $T_1 = \epsilon\mu$ . Then assuming that

trajectories are non-periodic which happens if

$$(0.7) \quad |V| \geq \epsilon_0,$$

$$(0.8) \quad |\nabla V| \geq \epsilon_0$$

we hope to get remainder estimate which is  $T_1^{-1}h = \mu^{-1}h$  times better than the principal part; since the latter is  $\asymp h^{-d}(1+\mu^r h^r)$  we expect to derive asymptotics with  $O(\mu^{-1}h^{1-d}(1+\mu^r h^r))$  remainder estimate under assumptions (0.7)-(0.8). Assuming sufficient smoothness (see below) it is a correct guess.

However without condition (0.8) trajectories could be periodic with period  $T_0 = \epsilon\mu^{-1}$  and we cannot expect remainder estimate better than  $O(\mu h^{1-d})$  as  $\mu h \leq 1$ .

On the other hand, if  $\text{rank } F < d$  we can follow trajectories until time  $T_1 = \epsilon$  and under assumptions (0.7)-(0.8) these trajectories are non-periodic. So, we expect  $O(h^{1-d}(1+\mu^r h^r))$  remainder estimate then.

However, even without condition (0.8) most of the trajectories are non-periodic in a short run due to the free movement. Actually, in the classical settings periodic trajectories must have  $\xi_{2r+1} = \dots = \xi_d = 0$  and thus form a set of measure 0 but we are in semiclassics and talking about non-periodicity one should be able to observe the shift after “quasi-period”  $\asymp \mu^{-1}$ . Anyway, we guess that the lack of condition (0.8) would not be that destructive (and even noticeable for not very large  $\mu$ ). Both guesses are correct under some smoothness assumption.

As  $\mu h \ll 1$  one can rescale to the *standard case* (i.e. with  $\mu = 1$ ). Really, after scaling  $\mu x \mapsto x$  and thus  $hD \mapsto \mu hD$ ,  $\mu \mapsto 1$ ,  $h \mapsto \mu h$  we get a standard case and scaling back we get the Weyl principal part

$$(0.9) \quad \mathcal{E}^W(x, x, \tau) = \omega_d(\tau - V)_+^{-\frac{d}{2}} h^{-d} \sqrt{g}$$

and the remainder estimate  $O(\mu h^{1-d})$ . Much more useful however is a rescaling applied to the intermediate rather than the final results.

### 0.3 Canonical form

Starting again from appropriate coordinates and applying  $h$ -Fourier transform with respect to  $x'' = (x_{r+1}, \dots, x_r)$  and change of variables  $x_j \mapsto x_j + \mu^{-1} f_j^{-1} \xi_{2j+1}$  with respect to  $x' = (x_1, \dots, x_r)$  we arrive instead of (0.5) to

$$(0.10) \quad \sum_{1 \leq j \leq r} (h^2 D_j^2 + \mu^2 f_j^2 x_j^2) + \sum_{1 \leq k \leq q} (h D_{2r+k})^2 + \\ V(x' + \mu^{-1} h F^{-1} D'', x'' - \mu^{-1} h F^{-1} D', x''')$$

where the third term is  $\mu^{-1}h$ -pseudo-differential operator.

Note that on bounded energy levels operators  $hD'$  and  $\mu x'$  are now bounded. Then one can apply Taylor decomposition to the last operator in (0.10) leading to an operator with the main part

$$(0.11) \quad \mathcal{A}^0 = \sum_{1 \leq j \leq r} (h^2 D_j^2 + \mu^2 f_j^2 x_j^2) + \sum_{1 \leq k \leq q} (h D_{2r+k})^2 + V(\mu^{-1} h F^{-1} D'', x'', x'''),$$

junior terms

$$(0.12) \quad \mathcal{A}' = \sum_{\alpha, \beta \in \mathbb{Z}^+, 1 \leq |\alpha| + |\beta| < l} W_{\alpha\beta}(\mu^{-1} h F^{-1} D'', x'', x''') \times (\mu^{-1} h)^{|\beta|} x'^{\alpha} D'^{\beta}$$

and the remainder estimate  $O(\mu^{-l})$  (subject to the smoothness assumptions).

Applying Hermite decomposition with respect to  $x'$  we see that as  $q = 0$  the main part  $\mathcal{A}^0$  becomes a family of  $\mu^{-1}h$ -pseudodifferential operators with respect to  $x''$

$$(0.13) \quad \mathcal{A}_\alpha = V_\alpha = \sum_{1 \leq k \leq r} (2\alpha_j + 1) f_j \mu h + V(\mu^{-1} h F^{-1} D'', x'', x'''),$$

while as  $q \geq 1$  it becomes a family of  $q$ -dimensional Schrödinger operators with respect to  $x'''$  with potentials which are  $\mu^{-1}h$ -pseudodifferential operators:

$$(0.14) \quad \begin{aligned} \mathcal{A}_\alpha &= \sum_{1 \leq k \leq q} (h D_{2r+k})^2 + V_\alpha, \\ V_\alpha &= \sum_{1 \leq k \leq r+q} (2\alpha_j + 1) f_j \mu h + V(\mu^{-1} h F^{-1} D'', x'', x'''). \end{aligned}$$

As  $q = 0$  the principal part of the spectral asymptotics for an individual operator is  $\asymp \mu^r h^{-r}$  with the remainder estimate  $O(\mu^{r-1} h^{1-r})$  under non-degeneracy condition

$$(0.15) \quad |V_\alpha| + |\nabla_{x'', \xi'''} V_\alpha| \geq \epsilon$$

and  $O(\mu^r h^{-r})$  otherwise (functions  $V_\alpha$  could be very flat). Assume that this non-degeneracy condition (which is equivalent to (0.8) as  $\mu h \ll 1$  and replaces it otherwise) holds. Then there are  $\asymp (\mu^{-r} h^{-r} + 1)$  contributing operators in the family (the rest is excluded due to ellipticity arguments) thus leading to the principal part and the remainder estimate  $\asymp h^{-d}$  and  $O(\mu^{-1} h^{1-2r})$  respectively as  $\mu h \leq 1$  and  $\asymp \mu^r h^{-r}$  and  $O(\mu^{r-1} h^{-r+1})$  as  $\mu h \geq 1$ .

For individual Schrödinger operator the principal part is  $\asymp \mu^r h^{r-d}$  with the best possible remainder estimate  $O(\mu^r h^{1+r-d})$  under condition (0.15). However this condition is not

crucial: this remainder estimate holds without such condition if either  $q \geq 3$  or  $q = 2$ ,  $V \in C^2$  and one can recover a weaker remainder estimate otherwise. Again there are  $\asymp (\mu^{-r}h^{-r} + 1)$  contributing operators in the family thus leading to the principal part and the remainder estimate  $\asymp h^{-d}$  and  $O(h^{1-2r})$  respectively as  $\mu h \leq 1$  and  $\asymp \mu^r h^{-r}$  and  $O(\mu^r h^{-r+1})$  as  $\mu h \geq 1$  (in the best case).

Surely one can take care of junior terms (0.12) and remove them if possible, thus reducing operator to Birkhoff normal form. It is easy in the smooth case as  $d = 2, 3$ ; as  $d = 2$  the normal form is

$$(0.16) \quad \mu^2 \sum_{m+j+k \geq 1} a_{mjk}(x_2, \mu^{-1}hD_2) \cdot (x_1^2 + \mu^{-2}h^2D_1^2)^m \mu^{-2m-2j-k} h^k$$

and for  $d = 3$  it is

$$(0.17) \quad \mu^2 \sum_{m+p+j+k \geq 1} a_{mpjk}(x_2, x_3, \mu^{-1}hD_2) \times \\ \times (x_1^2 + \mu^{-2}h^2D_1^2)^m (hD_3)^{2p} \mu^{-2m-2j-2p-k} h^k$$

but as  $r \geq 2$  the resonances become one of the obstacles: namely we cannot remove terms

$$(0.18) \quad a_{\alpha,\beta}(x'', x''', \mu^{-1}hD'') \times \prod_{1 \leq j \leq r} (x_j + i\mu^{-1}hD_j)^{\alpha_j} (x_j - i\mu^{-1}hD_j)^{\beta_j} \times (\xi''')^\gamma,$$

with

$$(0.19) \quad \sum_j (\alpha_j - \beta_j) f_j = 0$$

( $d = 3$  is a special case). So *resonance* means that  $\sum_j \gamma_j f_j = 0$  with  $\gamma \in \mathbb{Z}^r$ ;  $|\gamma|$  is an order of resonance.

As  $q = 0$ , the second order resonance terms are  $(x_j + i\mu^{-1}\xi_j)(x_k - i\mu^{-1}\xi_k)$  with  $f_j = f_k$ , the third order resonance terms are  $(x_j + i\mu^{-1}\xi_j)(x_k - i\mu^{-1}\xi_k)(x_m - i\mu^{-1}\xi_m)$  with  $f_j = f_k + f_m$  and their conjugates, etc. As  $q \geq 1$  there are additional (but less malicious in the end of the day) resonance terms.

Another obstacle is the lack of a very large smoothness. I overcome both of these obstacles because most of the junior terms are non-essential small perturbations: as  $q \geq 2$  all junior terms are non-essential; as  $q = 1$  all terms but of the form  $\mu^{-1}(x_j + i\mu^{-1}\xi_j)(x_k - i\mu^{-1}\xi_k)(x_m - i\mu^{-1}\xi_m)$  are also non-essential (but such terms do not appear for special operator (0.2)), while for  $q = 0$  there are more essential terms.

Further, the canonical form reduction is possible for general operator (0.1) as well but there will be essential cubic perturbations and third-order resonances become important as  $q = 1$ .

## 0.4 Main tools and classification

We will consider  $u(x, y, t)$  the Schwartz kernel of operator  $U(t) = e^{ih^{-1}t\tilde{A}}$  where  $\tilde{A}$  is one of two framing approximations for  $A$ ; in the case of irregular coefficients  $\tilde{A}$  will include mollification as well. However even in the case of smooth coefficients it is often convenient to take  $\tilde{A}$  different from  $A$ .

Microlocal analysis (propagation of singularities) of  $u(x, y, t)$  in its original form or transformed (as  $A$  is reduced to its canonical form) plays a crucial role. The former approach is used in the case of *weak* magnetic field and it is combined with the standard theory rescaled. In the latter approach is used in the case of *intermediate* and stronger magnetic field and it is combined with the successive approximation method to construct solutions.

As  $q = 0$  and non-degeneracy condition holds we need to take mollification parameter  $\varepsilon = C\mu h |\log h|$  to make a weak magnetic field approach working properly. On the other hand, if we reduce operator to its canonical form, we must take  $\varepsilon = C(\mu^{-1}h |\log \mu|)^{\frac{1}{2}}$  as  $d = 2$  and  $\varepsilon = C \max(\mu^{-1}, (\mu^{-1}h |\log \mu|)^{\frac{1}{2}})$  as  $d \geq 4$ . Since we are interested in the smallest possible  $\varepsilon$  we apply weak magnetic field approach for  $\mu \leq C(h |\log h|)^{-\frac{1}{3}}$  as  $d = 2$  and for  $\mu \leq C(h |\log h|)^{-\frac{1}{2}}$  as  $d \geq 2$ . Otherwise we refer to *intermediate* magnetic field case as  $\mu \leq C(h |\log h|)^{-1}$ , then *strong* as  $\mu \leq \epsilon h^{-1}$ , *superstrong* as  $\mu \leq Ch^{-1}$  and *ultrastrong* as  $\mu \geq Ch^{-1}$ .

This classification holds as  $q \geq 1$  but with a twist. If magnetic field is weak we study only “original” propagator  $u(x, y, t)$  picking up  $\varepsilon = C\rho^{-1}h |\log h|$  with  $\rho = |\xi'''|$  as  $\rho \geq \bar{\rho}_1 = C \max(\mu^{-1}, (\mu h |\log h|)^{\frac{1}{2}})$  and  $\varepsilon = C\bar{\rho}_1^{-1}h |\log h|$  otherwise. Inner zone  $\{|\xi'''| \leq \bar{\rho}_1\}$  is treated by referring to the standard results rescaled.

If magnetic field is intermediate we apply a weak magnetic field approach in the outer zone  $\{|\xi'''| \geq \bar{\rho}_1\}$ . In the inner zone we apply reduction to the canonical form. Threshold between weak and intermediate magnetic field cases depends on  $q$  and the presence of non-degeneracy condition.

In the strong (and stronger) magnetic field cases we use canonical form reduction everywhere and  $\varepsilon = Ch |\log h|$ .

Actually the above choice of  $\varepsilon$  is needed to provide a reduction; the further analysis as  $q \geq 1$  requires to increase  $\varepsilon$  and the actual choice of  $\varepsilon$  varies (as  $q = 1, 2$ ) depending on other assumptions and those indicated are kind of milestones rather than the actual values.

## 1 Weak Magnetic Field

We assume that magnetic field is relatively weak and we prove our results by reducing to the standard case.

## 1.1 Standard reference result rescaled

As I mentioned the intermediate result in the standard case will be valuable for us more than the final one; namely we have by rescaling

$$(1.1) \quad |F_{t \rightarrow h^{-1}\tau}(\bar{\chi}_T(t)\Gamma_x u) - \int T\hat{\chi}((\tau - \lambda)\frac{T}{h})d\lambda \mathcal{E}^{\text{MW}}(x, \lambda)| \leq Ch^{3-d} \quad \forall \tau : |\tau| \leq \epsilon$$

as  $\mu \leq \epsilon h^{-1}|\log h|^{-1}$ ,  $Ch|\log h| \leq T \leq \bar{T}_0 = \epsilon\mu^{-1}$ ; here and below  $\bar{\chi}$  is smooth and even satisfying admissibility condition of [BrIvr] function, supported in  $[-1, 1]$  and equal 1 in  $[-\frac{1}{2}, \frac{1}{2}]$ ,  $\hat{\chi}$  is its Fourier transform,  $\Gamma_x v = v(x, x, t)$ .

We use also notations  $\Gamma v = \int \Gamma_x v dx$  and  $\phi_T(t) = \phi(t/T)$ . Also we need to assume condition (0.7). Estimate (1.1) holds as coefficients of operator are either smooth or mollified with  $\varepsilon \geq Ch|\log h|$ .

## 1.2 Full-rank case

We want to increase  $T$  in (1.1). To do this we analyze propagation of singularities. Let us introduce *slow variables*

$$(1.2) \quad X_j = x_j - \sum \beta_{jk} p_k(x, \xi), \quad p_k(x, \xi) = \mu^{-1} \xi_k - V_k$$

with  $(\beta_{jk})$  inverse matrix to  $(F_{jk})$ .

Then the Poisson brackets satisfy

$$(1.3) \quad \{p_k, X_j\} = 0 \quad \forall j, k,$$

$$(1.4) \quad \{X_j, X_k\} = \mu^{-1} \beta_{jk}, \quad \{p_j, p_k\} = -\mu^{-1} h F_{jk} \quad \forall j, k.$$

Then in the classical dynamics for time  $T$  the shift of  $\phi(x, \xi) = \sum_j \ell_j X_j$  is equal to  $\mu^{-1} T (\sum_{j,k} \beta_{jk} \ell_j \partial_k V + o(1))$  as  $T \leq \bar{T}_1 = \epsilon\mu$  and under condition (0.8) it will be of magnitude  $\mu^{-1} T$  for an appropriate vector  $\ell$ ,  $|\ell| = 1$ .

To make it observable from the point of view of microlocal analysis one needs to satisfy the *logarithmic uncertainty principle* (see [BrIvr]) which in this case is

$$(1.5) \quad \mu^{-1} T \times \varepsilon \geq C \mu^{-1} h |\log h|$$

because  $X_1, \dots, X_d$  are linked by (1.5) and semiclassical parameter is therefore  $\mu^{-1} h$ . We can avoid condition

$$(1.6) \quad \varepsilon \times \varepsilon \geq C \mu^{-1} h |\log h|$$



because we do not use any reduction in this case; condition (1.6) be very unpleasant for not very large  $\mu$ .

We want to satisfy (1.6) with  $T = \bar{T}_0$  and therefore we pick up the smallest  $\varepsilon$  to satisfy this condition:

$$(1.7) \quad \varepsilon = C\mu h |\log h|.$$

Then we prove that

$$(1.8) \quad |F_{t \rightarrow h^{-1}\tau} \chi_T(t) \Gamma(u\psi)| \leq Ch^s \quad \forall \tau : |\tau| \leq \epsilon \quad \forall T \in [\bar{T}_0, \bar{T}_1]$$

where  $\psi = \psi(x)$  is a cut-off function described above,  $\chi$  is smooth (and [BrIvr] type) function supported in  $[-1, -\frac{1}{2}] \cup [\frac{1}{2}, 1]$ .

Then (1.1) and (1.8) imply that estimate

$$(1.9) \quad |F_{t \rightarrow h^{-1}\tau} \bar{\chi}_T(t) \Gamma(u\psi) - \int \int T \hat{\chi}((\tau - \lambda) \frac{T}{h}) d_\lambda \mathcal{E}^{\text{MW}}(x, \lambda) \psi(x) dx| \leq Ch^{3-d}$$

holds with any  $T \in [\bar{T}_0, \bar{T}_1]$  and  $\forall \tau : |\tau| \leq \epsilon$ .

We must here and below integrate with respect to  $\psi(x) dx$  because  $X_j$  contain  $\xi_k$  and we need a real trace  $\Gamma$  and not just a restriction to the diagonal  $\Gamma_x$ .

Then Tauberian arguments imply spectral asymptotics with the remainder estimate  $O(\bar{T}_1^{-1} h^{1-d}) = O(\mu^{-1} h^{1-d})$  while an approximation error is  $O(\varepsilon^l |\log \varepsilon|^{-\sigma} h^{-d})$  provided

$$(1.10) \quad V \in C^{l,\sigma} \quad (l, \sigma) \succeq (1, 1)$$

where  $C^{l,\sigma}$  is a class of functions with  $k$ -th derivatives continuous with continuity modulus  $t^{l-k} |\log t|^{-\sigma}$  functions,  $k = \lfloor l \rfloor$  unless  $l \in \mathbb{Z}$ ,  $\sigma \leq 0$  when  $k = l - 1$ .

Thus we arrive to our first statement:

**Theorem 1.1.** *Let  $d = 2r$ ,  $V \in C^{l,\sigma}$  with  $(l, \sigma) \succeq (1, 2)$  and conditions (0.7), (0.8) hold. Then for*

$$(1.11) \quad \mu \leq h^{\delta-1}$$

*with an arbitrarily small exponent  $\delta > 0$  the following estimate holds*

$$(1.12) \quad \mathcal{R} \stackrel{\text{def}}{=} \left| \int (e(x, x, 0) - \mathcal{E}^{\text{MW}}(x, 0)) \psi(x) dx \right| \leq C\mu^{-1} h^{1-d} + C(\mu h |\log h|)^l |\log h|^{-\sigma} h^{-d}.$$

*In particular, as  $(l, \sigma) = (3, 1)$  sharp remainder estimate  $\mathcal{R} \leq C\mu^{-1} h^{1-d}$  holds for  $\mu \leq h^{-\frac{1}{2}} |\log h|^{-\frac{1}{2}}$  and as  $(l, \sigma) = (2, 1)$  this sharp remainder estimate holds for  $\mu \leq h^{-\frac{1}{3}} |\log h|^{-\frac{1}{3}}$ .*

### 1.3 Non-full-rank case

As  $q \geq 1$  we can use a free movement along  $x'''$  to extend (1.9) from  $T = \bar{T}_0$  to larger  $T$ . Let us introduce  $\rho$ -partition with respect to  $\xi'''$  with  $\rho = \frac{1}{2}|\xi'''|$  and let us consider a partition element with  $\rho \geq C\mu^{-1}$ . Then for time  $T \leq \bar{T}_1 = \epsilon\rho$  the shift with respect to  $x'''$  will be of magnitude  $\epsilon\rho$  and in order to be observable it must satisfy logarithmic uncertainty principle

$$(1.13) \quad \rho \times \rho T \geq Ch|\log h| \iff T \geq C\rho^{-2}h|\log h|.$$

In order to be able to plug  $T = \bar{T}_0 = \epsilon\mu^{-1}$  we must take

$$(1.14) \quad \rho \geq \bar{\rho}_1 = C \max(\mu^{-1}, (\mu h|\log h|)^{\frac{1}{2}}).$$

Then both (1.8) and (1.9) hold for  $u\psi$  replaced by  $Q(hD_x)(u\psi)$  with  $h$ -pseudodifferential operator  $Q$  with symbol supported in the indicated partition element.

Then due to Tauberian arguments the contribution of this element to the remainder estimate is  $Ch^{1-d}\bar{T}_1^{-1}\rho^q$  and therefore the total contribution of the *outer* zone  $\{|\xi'''| \geq \bar{\rho}_1\}$  to the remainder estimate does not exceed  $Ch^{1-d} \int \bar{T}_1^{-1}(\rho)\rho^{q-1} d\rho$  which is  $O(h^{1-d})$  as  $q \geq 2$  and  $O(h^{1-d}\log \mu)$  as  $q = 1$ . To get rid off the logarithmic factor we prove that for  $(l, \sigma) \succeq (1, 2)$  one can take  $\bar{T}_1 = \epsilon\rho|\log \rho|^2$  in the propagation.

Also in this zone  $\varepsilon$  must satisfy logarithmic uncertainty principle  $\varepsilon \times \rho \geq Ch|\log h|$  and taking  $\varepsilon = C\rho^{-1}h|\log h|$  there we get mollification error  $Ch^{-d} \int \varepsilon^l |\log \varepsilon|^{-\sigma} \rho^{q-1} d\rho$  which is  $O(h^{1-d})$  as either  $q \geq 2$ ,  $(l, \sigma) \succeq (1, 1)$  or  $q = 1$ ,  $(l, \sigma) \succeq (1, 2)$ . Note that now mollification parameter  $\varepsilon$  depends on  $(x, \xi)$ .

On the other hand, contribution of the zone  $\{|\xi'''| \leq \bar{\rho}_1\}$  to the remainder estimate is  $C\bar{T}_0^{-1}\bar{\rho}_1^q h^{1-d} = O(h^{1-d} + (\mu h)^{\frac{q}{2}+1}|\log h|^{\frac{q}{2}}h^{-d})$ .

Finally, in this zone we pick  $\varepsilon = C\bar{\rho}_1^{-1}h|\log h|$  and its contribution to the mollification error will be less than what we already got.

So we arrive to estimate

$$(1.15) \quad \mathcal{R} \leq Ch^{1-d} + C(\mu h)^{\frac{q}{2}+1}|\log h|^{\frac{q}{2}}h^{-d}$$

which actually could be improved to

**Theorem 1.2.** *Let condition (0.7) be fulfilled. Let either  $q = 1$ ,  $V \in C^{1,2}$  or  $q \geq 2$ ,  $V \in C^{1,1}$ . Then*

$$(1.16) \quad \mathcal{R} \leq Ch^{1-d} + C(\mu h)^{\frac{q}{2}+1}h^{-d}.$$

*In particular, for  $\mu \leq \mu_{1(q)}^* = h^{-\frac{q}{q+2}}$  sharp remainder estimate  $\mathcal{R} \leq Ch^{1-d}$  holds. As  $q \geq 2$  this sharp remainder estimate holds for  $\mu \leq h^{-\frac{1}{2}}$ .*

Note, that at this stage extra smoothness is not very useful. Furthermore we can assume in what follows that  $\mu \geq \mu_{1(q)}^*$  and thus  $\bar{\rho}_1 = C(\mu h |\log h|)^{\frac{1}{2}}$ .

To improve estimates (1.15), (1.16) one needs to use better arguments in the inner zone  $\mathcal{Z} = \{|\xi'''| \leq \bar{\rho}_1\}$ . Under non-degeneracy condition (0.8) we can apply the same arguments as in the proof of theorem 1.1 as long as  $\varepsilon = C\mu h |\log h|$  in  $\mathcal{Z}$ . Then contribution of this zone to the remainder estimate becomes  $O(h^{1-d})$  while its contribution to an approximation error becomes  $O(\bar{\rho}_1^q \varepsilon^l |\log \varepsilon|^{-\sigma} h^{-d})$  and we arrive to

**Theorem 1.3.** *Let condition (0.7), (0.8) be fulfilled. Let  $q \geq 1$ ,  $V \in C^{l,\sigma}$  and either  $q = 1$ ,  $(l, \sigma) \succeq (1, 2)$  or  $q \geq 2$ ,  $(l, \sigma) \succeq (1, 1)$ . Then*

$$(1.17) \quad \mathcal{R} \leq Ch^{1-d} + C(\mu h)^{\frac{q}{2}+l} |\log h|^{l-\sigma} h^{-d}.$$

*In particular, sharp remainder estimate  $\mathcal{R} \leq Ch^{1-d}$  holds for  $\mu \leq h^{-\frac{1}{2}} |\log h|^{-\frac{1}{2}}$  as  $q = 1$ ,  $(l, \sigma) = (\frac{3}{2}, \frac{1}{2})$ .*

## 2 Intermediate and strong magnetic field

Now magnetic field is stronger than before but still either below  $\epsilon h^{-1} |\log h|^{-1}$  or between this value and  $\epsilon h^{-1}$ . There is certain difference in the analysis of these two cases and for general operator (0.1) some statements would slightly differ as well.

### 2.1 Full-rank case

In this case we pick

$$(2.1) \quad \varepsilon = \begin{cases} C(\mu^{-1} h |\log h|)^{\frac{1}{2}} & r = 1, \\ C \max(\mu^{-1}, (\mu^{-1} h |\log h|)^{\frac{1}{2}}) & r \geq 2 \end{cases}$$

and reduce operator to a canonical form in the smooth case or “a poor man’s canonical form” otherwise. Cases  $r = 1$  and  $r \geq 2$  differ because of the reduction: we need to solve equation

$$(2.2) \quad \{a_0, S\} = V - W, \quad a_0 = \sum_{1 \leq j \leq r} f_j(x_j^2 + \xi_j^2)$$

where  $W$  consists of unremovable terms; as  $r = 1$  this equation is solved by integration along circles leaving  $W = W(x_2, \xi_2, (x_1^2 + \xi_1^2)^{\frac{1}{2}})$  while for  $r \geq 2$  it is solved by Taylor

decomposition and we must assume that  $\varepsilon \geq C\mu^{-1}$ . On the other hand, we always need to assume (1.6) now because we consider  $x'', \xi''$  as dual variable and we need to consider  $\mu^{-1}h$ -pseudodifferential operators.

In the best possible case we would get something similar the family of separate scalar  $\mu^{-1}h$ -pseudodifferential operators but the same results hold in the general case as well:

**Theorem 2.1.** *Let  $d = 2r$ ,  $V \in C^{l,\sigma}$  with  $(l, \sigma) \succeq (1, 2)$  and conditions (0.7), (0.8) hold. Then*

(i) *For  $\mu \leq h^{-1}|\log h|^{-1}$  estimate*

$$(2.3) \quad \mathcal{R} \leq C\mu^{-1}h^{1-d} + C\varepsilon^l |\log h|^{-\sigma} h^{-d} \quad \forall \tau : |\tau| \leq \epsilon$$

*holds with  $\varepsilon = \mu^{-1}$ ;*

(ii) *For  $h^{-1}|\log h|^{-1} \leq \mu \leq \epsilon h^{-1}$  estimate (2.3) holds with  $\varepsilon = C(\mu^{-1}h|\log h|)^{\frac{1}{2}}$ .*

*In particular, as  $(l, \sigma) = (3, 1)$  sharp remainder estimate estimate  $\mathcal{R} \leq C\mu^{-1}h^{1-d}$  holds for  $h^{-\frac{1}{2}}|\log h|^{-\frac{1}{2}} \leq \mu \leq \epsilon h^{-1}$ ;*

(iii) *As  $r = 1$ , estimate (2.3) holds with  $\varepsilon = (\mu^{-1}h|\log h|)^{\frac{1}{2}}$  but for  $\mathcal{R}_I$  instead of  $\mathcal{R}$ ; here and below  $\mathcal{R}_I$  is defined by formula (1.12) but with  $\mathcal{E}^{\text{MW}}$  replaced by  $\mathcal{E}_I^{\text{MW}}$  which is defined by (0.6) with  $V(x)$  replaced by  $W(x)$  where  $W(x)$  is an average of  $V(y)$  along circle  $C_x = \{y : |x - y| = (\mu f)^{-1}(\tau - V(x))^{\frac{1}{2}}\}$ ;  $f$  is a scalar intensity of magnetic field.*

*In particular, as  $(l, \sigma) = (2, 1)$  sharp remainder estimate estimate  $\mathcal{R}_I \leq C\mu^{-1}h^{1-d}$  holds for  $\mu \geq h^{-\frac{1}{3}}|\log h|^{-\frac{1}{3}}$ .*

This statement together with theorem 1.1 cover case  $\mu \leq \epsilon h^{-1}$  completely.

## 2.2 Non-full rank case. I

The same classification and definition of  $\varepsilon$  persist as  $q \geq 1$ ; however we reduce operator to a canonical form only in inner zone  $\mathcal{Z} = \{|\xi'''| \leq \bar{\rho}_1\}$  (in the intermediate magnetic field case); in the outer zone we apply the weak magnetic field approach and pick up  $\varepsilon = C|\xi'''|^{-1}h|\log h|$ ; furthermore after reduction is done,  $\varepsilon$  is redefined (increased) in the inner zone as well.

There are few different statements to prove; the first one is a generic one:

**Theorem 2.2.** *Let  $q \geq 1$ ,  $V \in C^{l,\sigma}$ . Then*

(i) *As  $q \geq 3$ ,  $(l, \sigma) = (1, 1)$  sharp remainder estimate  $\mathcal{R} \leq Ch^{1-d}$  holds for  $\mu \leq Ch^{-1}$ ;*

(ii) *As  $q = 2$ ,  $(l, \sigma) = (1, 1)$  remainder estimate*

$$(2.4) \quad \mathcal{R} \leq Ch^{1-d} + C\mu h^{\frac{5}{3}-d}$$

holds for  $\mu \leq Ch^{-1}$ ;

(iii) As  $q = 1$ ,  $(l, \sigma) \succeq (1, 2)$  remainder estimate

$$(2.5) \quad \mathcal{R} \leq C\mu h^{\frac{4}{3}-d} + C\mu^{1-\frac{l}{2}}h^{1-d}|\log h|^{-\frac{\sigma}{2}}$$

holds for  $h^{-\frac{1}{2}}|\log h|^{-\frac{1}{2}} \leq \mu \leq Ch^{-1}$ ;

(iv) As  $d = 3$ ,  $(l, \sigma) = (1, 2)$  remainder estimate

$$(2.6) \quad \mathcal{R}_I \leq C\mu h^{\frac{4}{3}-d}$$

holds for  $h^{-\frac{1}{3}} \leq \mu \leq Ch^{-1}$ ; here  $\mathcal{R}_I$  is again defined by (1.12) with  $\mathcal{E}^{\text{MW}}$  replaced by  $\mathcal{E}_I^{\text{MW}}$  which is defined by (0.6) with  $V(x)$  replaced by  $W(x)$  where  $W(x)$  is an average of  $V(y)$  along circle  $\mathcal{C}_x = \{y : x_3 = y_3, |x - y| = (\mu f)^{-1}(\tau - V(x))^{\frac{1}{2}}\}$  as magnetic field is directed along  $x_3$ .

In what follows we need to treat only cases  $q = 1, 2$ . As non-degeneracy condition is fulfilled we get

**Theorem 2.3.** *Let  $q = 1, 2$ ,  $V \in C^{l, \sigma}$  and conditions (0.7), (0.8) hold. Then*

(i) As  $q = 2$ ,  $(l, \sigma) = (1, 1)$  sharp remainder estimate  $\mathcal{R} \leq Ch^{1-d}$  holds for  $\mu \leq Ch^{-1}$ ;

(ii) As  $q = 1$ ,  $(l, \sigma) \succeq (1, 2)$  remainder estimate

$$(2.7) \quad \mathcal{R} \leq Ch^{1-d} + C\mu^{\frac{1}{2}-l}h^{\frac{1}{2}-d}|\log h|^{\frac{1}{2}-\sigma}$$

holds for  $h^{-\frac{1}{2}}|\log h|^{-\frac{1}{2}} \leq \mu \leq Ch^{-1}$ ;

(iii) As  $d = 3$ ,  $(l, \sigma) = (1, 2)$  sharp remainder estimate  $\mathcal{R}_I \leq Ch^{1-d}$  holds for  $\mu \leq Ch^{-1}$ .

So far all the the results of the article could be generalized to a general operator (0.1) with modification of non-degeneracy condition (0.8) and, as  $q = 0, 1$ ,  $r \geq 2$  with the special attention to the third order resonances (because in the general case they could lead to non-removable  $O(\mu^{-1})$  terms in the canonical form).

## 2.3 Non-full-rank case. II

In this section we exploit more specific properties of operator (0.2), namely that  $f_j$  have constant multiplicities.

**Theorem 2.4.** *Let  $q = 1, 2$ ,  $V \in C^{l, \sigma}$  and condition (0.7) hold. Then*

(i) As  $q = 2$ ,  $(l, \sigma) \succeq (1, 1)$  remainder estimate

$$(2.8) \quad \mathcal{R} \leq Ch^{1-d} + C\mu h^{\frac{2l}{l+2}+1-d}|\log h|^{-\frac{2\sigma}{l+2}}$$

holds for  $\mu \leq Ch^{-1}$ ; in particular sharp remainder estimate  $\mathcal{R} \leq Ch^{1-d}$  holds as  $(l, \sigma) = (2, 0)$ .

(ii) As  $q = 1$ ,  $(1, 2) \preceq (l, \sigma) \preceq (2, 0)$  remainder estimate

$$(2.9) \quad \mathcal{R} \leq Ch^{1-d} + C\mu h^{\frac{l}{l+2}+1-d} |\log h|^{-\frac{\sigma}{l+2}} + C\mu^{1-\frac{l}{2}} h^{1-d} |\log h|^{-\frac{\sigma}{2}}$$

holds for  $h^{-\frac{1}{2}} |\log h|^{-\frac{1}{2}} \leq \mu \leq Ch^{-1}$ ;

(iii) As  $d = 3$ ,  $(1, 2) \preceq (l, \sigma) \preceq (2, 0)$  remainder estimate

$$(2.10) \quad \mathcal{R}_I \leq Ch^{1-d} + C\mu h^{\frac{l}{l+2}+1-d} |\log h|^{-\frac{\sigma}{l+2}}$$

holds for  $h^{-\frac{1}{3}} |\log h|^{-\frac{1}{3}} \leq \mu \leq Ch^{-1}$ .

Further, as  $r \geq 2$  Diophantine properties of  $(f_1, \dots, f_r)$  can play role. Assume that

$$(2.11) \quad \mathbf{n}(\hbar, \tau) \stackrel{\text{def}}{=} \#\{\alpha \in \mathbb{Z}^{+r}, \sum_j (2\alpha_j + 1)f_j + V\hbar < \tau\}$$

satisfies estimate

$$(2.12) \quad |\mathbf{n}(\hbar, \tau) - \mathbf{n}(\hbar, \tau')| \leq C\hbar^{-r} (|\tau - \tau'| + \nu(\hbar))$$

$$\forall \hbar \in (0, 1] \forall \tau, \tau' : |\tau| \leq C, |\tau'| \leq C$$

with  $\nu(\hbar) = o(\hbar)$  (it holds with  $\nu(\hbar) = \hbar$  for sure). Two following theorems improve theorems 2.2, 2.4 respectively:

**Theorem 2.5.** *Let  $r \geq 2$ ,  $q = 1, 2$ ,  $V \in C^{l, \sigma}$  and conditions (0.7), (2.11) hold. Then*

(i) As  $q = 2$ ,  $(l, \sigma) \succeq (1, 1)$  remainder estimate

$$(2.13) \quad \mathcal{R} \leq Ch^{1-d} + C\nu(\mu h)h^{\frac{2}{3}-d}$$

holds for  $\mu \leq Ch^{-1}$ ;

(ii) As  $q = 1$ ,  $(l, \sigma) \succeq (1, 2)$  remainder estimate

$$(2.14) \quad \mathcal{R} \leq Ch^{1-d} + C\mu^{\frac{1}{2}-l} h^{\frac{1}{2}-d} |\log h|^{\frac{1}{2}-\sigma} +$$

$$C\nu(\mu h) \left( h^{\frac{1}{3}-d} + \mu^{-\frac{l}{2}} h^{-d} |\log h|^{-\frac{\sigma}{2}} \right)$$

holds for  $\mu \leq Ch^{-1}$ .

**Theorem 2.6.** *Let  $r \geq 2$ ,  $q = 1, 2$ ,  $V \in C^{l,\sigma}$  and conditions (0.7), (2.11) hold. Then*

(i) *As  $q = 2$ ,  $(l, \sigma) \succeq (1, 1)$  remainder estimate*

$$(2.15) \quad \mathcal{R} \leq Ch^{1-d} + C\nu(\mu h)h^{\frac{2l}{l+2}-d}|\log h|^{-\frac{2\sigma}{l+2}}$$

*holds for  $\mu \leq Ch^{-1}$ ;*

(ii) *As  $q = 1$ ,  $(l, \sigma) \succeq (1, 2)$  remainder estimate*

$$(2.16) \quad \mathcal{R} \leq Ch^{1-d} + C\mu^{\frac{1}{2}-l}h^{\frac{1}{2}-d}|\log h|^{\frac{1}{2}-\sigma} + \\ C\nu(\mu h)\left(h^{\frac{l}{l+2}-d}|\log h|^{-\frac{\sigma}{l+2}} + \mu^{-\frac{l}{2}}h^{-d}|\log h|^{-\frac{\sigma}{2}}\right)$$

*holds for  $\mu \leq Ch^{-1}$ .*

Note that in estimates (2.13)–(2.16) right-hand expressions are sums of the right-hand expressions under non-degeneracy condition and of the right-hand expressions without microhyperbolicity conditions, but latter are multiplied by  $\nu(\mu h)(\mu h)^{-1}$ .

### 3 Superstrong and ultrastrong magnetic field

In this case  $\mu \geq \epsilon h^{-1}$  and the distance Landau levels increases; in the case of the ultrastrong magnetic field only one (may be multiple) level should be considered.

#### 3.1 Superstrong magnetic field

In this case  $h^{-1}\epsilon \leq \mu \leq Ch^{-1}$  and the magnitude of the principal part of the asymptotics is still the same ( $h^{-d}$ ) as well as the remainder estimates in theorems 2.1–2.4. The only difference is that non-degeneracy condition (0.8) is replaced by

$$(3.1) \quad |\tau - V - \sum_j (2\alpha_j + 1)f_j\mu h| + |\nabla V| \geq \epsilon_0 \quad \forall \alpha \in \mathbb{Z}^{+r}.$$

**Theorem 3.1.** (i) *Statements of theorems 2.2, 2.4 remain true for  $h^{-1}\epsilon \leq \mu \leq Ch^{-1}$ ;*

(ii) *Statements of theorems 2.1, 2.3 remain true for  $h^{-1}\epsilon \leq \mu \leq Ch^{-1}$  with condition (0.8) replaced by (3.1) with  $\tau = 0$  and condition (0.7) skipped;*

(iii) *As  $q = 0$  under condition*

$$(3.2) \quad |\tau - V - \sum_j (2\alpha_j + 1)f_j\mu h| \geq \epsilon_0 \quad \forall \alpha \in \mathbb{Z}^{+r}$$

*with  $\tau = 0$  estimate  $\mathcal{R} \leq Ch^s$  holds with arbitrarily large  $s$  (spectral gaps).*

**Remark 3.2.** As  $1 \leq \mu \leq Ch^{-1}$  under condition

$$(3.3) \quad \tau - V - \sum_j f_j \mu h \leq -\epsilon_0$$

estimate  $\mathcal{R} \leq Ch^s$  holds with  $\mathcal{E}^{\text{MW}}(x, \tau) = 0$  ( $\tau = 0$ ) and with arbitrarily large  $s$ .

### 3.2 Ultrastrong magnetic field

In this case  $\mu \geq Ch^{-1}$  and in order not to be below the bottom of the spectrum one should modify condition  $V \in C^{l, \sigma}$ . Assume instead that

$$(3.4) \quad V = - \sum_j (2\bar{\alpha}_j + 1) f_j \mu h + W, \quad \bar{\alpha} \in \mathbb{Z}^{+r}, \quad W \in C^{l, \sigma} \quad (q = 0);$$

$$(3.5) \quad V = - \sum_j f_j \mu h + W, \quad W \in C^{l, \sigma} \quad (q \geq 1).$$

**Theorem 3.3.** *Let  $q = 0$  and conditions (3.4), (3.1) be fulfilled with  $\tau = 0$ . Then*

(i) *Estimate*

$$(3.6) \quad \mathcal{R} \leq C\mu^{r-1}h^{-r+1};$$

*holds for  $\mu \geq Ch^{-1}$ ;*

(ii) *Furthermore, under condition (3.2) estimate  $\mathcal{R} \leq C\mu^{-s}$  holds with arbitrarily large  $s$ .*

**Theorem 3.4.** *Let  $q \geq 1$  and condition (3.5) be fulfilled. Then*

(i) *Estimate*

$$(3.7) \quad \mathcal{R} \leq C\mu^r h^{-r+1}$$

*holds for  $\mu \geq Ch^{-1}$ ,  $q \geq 3$ ;*

(ii) *Estimate*

$$(3.8) \quad \mathcal{R} \leq C\mu^r h^{-r+1} (1 + h^{-1+\frac{lq}{l+2}} |\log h|^{-\frac{\sigma q}{l+2}})$$

*holds for  $\mu \geq Ch^{-1}$ ,  $q = 1, 2$ ; in particular, as  $q = 2$ ,  $(l, \sigma) = (2, 0)$  sharp remainder estimate (3.7) holds.*

(iii) *Under condition (3.1) sharp remainder estimate (3.7) holds for  $\mu \geq Ch^{-1}$ ,  $q = 1, 2$ .*

(iv) *Under condition (3.3) estimate  $\mathcal{R} \leq C\mu^r h^s$  holds with  $\mathcal{E}^{\text{MW}}(x, \tau) = 0$  ( $\tau = 0$ ) and with arbitrarily large  $s$ .*



## 4 Remarks. Generalizations

One can generalize the results stated above.

**Remark 4.1.** One can get rid of condition (0.7) by method of rescaling; then all the results remain the same.

**Remark 4.2.** Instead of operator (0.2) one can consider operator (0.1) in its full generality:

- (i) as  $q \geq 3$  no modification in conditions is needed.
- (ii) As  $q = 0, 1, 2$  non-degeneracy conditions (0.8) and (3.1) should be modified; as  $r = 1$  one needs to replace  $|\nabla V|$  by  $|\nabla(V/f)|$  with  $f = f_1$  ( $\tau = 0$ ); as  $r \geq 2$  the modification is more profound because we are essentially in the matrix situation. For example if  $f_1, \dots, f_d$  have constant multiplicities our condition looks like

$$(4.1) \quad |V + \sum_j f_j \tau_j| + |\nabla(V^{-1}(\sum_j f_j \tau_j))| \geq \epsilon_0 \quad \forall \tau_1 \geq 0, \dots, \tau_r \geq 0$$

as ( $\tau = 0$ ) and we need to assume that (0.7) holds.

- (iii) Smoothness conditions to  $g^{jk}$ ,  $F_{jk}$  should be at least  $(l, \sigma)$  and also at least  $(2, 1)$  (required for reduction arguments) but for they could be even stronger to get a proper mollification error.

(iv) Constant multiplicity of  $f_j$  which was taken for granted is no more guaranteed and the results of subsection 2.3 require it.

(v) Further, condition (2.12) should be fulfilled for  $\mathbf{n}(x, \hbar)$  integrated with respect to  $x$  and it is fulfilled automatically with  $\nu(\hbar) = \hbar^\kappa$  provided  $\{\nabla(f_2/f_1), \dots, \nabla(f_r/f_1)\}$  has at least rank  $\kappa - 1$ .

(vi) As  $q = 1$  and non-degeneracy condition is not fulfilled, unremovable  $O(\mu^{-1})$  terms in the canonical form could lead to a some correction term in order to save the estimate. These terms can appear due to variable  $g^{jk}$ ,  $F_{jk}$  even if  $f_j = \text{const}$  and they are unremovable due to the third-order resonances.

(vii) In the case of  $q = 0$  and the ultrastrong magnetic field one should require that for each  $\alpha \in \mathbb{Z}^{+r}$  such that  $|\sum_j (\bar{\alpha}_j - \alpha_j) f_j| \leq \epsilon$  for each  $j$  either  $\bar{\alpha}_j = \alpha_j$  or  $f_j$  has a constant multiplicity.

**Remark 4.3.** (i) The results of this article are proven in three papers [Ivr3], [Ivr4], [Ivr5] where the first one is dealing with 2,3-dimensional cases and the second and the third are dealing with the higher dimensions. One can access them from

<http://www.math.toronto.edu:/ivrii/Research/preprints.html>

as well as relevant talks to be viewed on a computer screen rather than printed.

(ii) In my forthcoming papers I am planning to get rid off assumption “rank( $F_{jk}$ ) = const”; surely results of the case  $q = 0$  will be no more valid.

## References

- [BrIvr] M. BRONSTEIN, V. IVRII. *Sharp Spectral Asymptotics for Operators with Irregular Coefficients. Pushing the Limits*, Comm. Partial Differential Equations, **28** (2003) 1&2, 99–123.
- [Dim] M. DIMASSI. *Développements asymptotiques de l'opérateur de Schrödinger avec champ magnétique fort*, Comm. Partial Differential Equations, **26** (2001) 3&4, 595–627.
- [Ivr1] V. IVRII. *Microlocal Analysis and Precise Spectral Asymptotics*, Springer-Verlag, SMM, 1998, xv+731.
- [Ivr2] V. IVRII. *Sharp Spectral Asymptotics for operators with irregular coefficients. II. Boundary and Degenerations*, Comm. Partial Differential Equations, **28** (2003) 1&2, 125–156.
- [Ivr3] V. IVRII. *Sharp spectral asymptotics for operators with irregular coefficients. III Schrödinger operator with a strong magnetic field*, 79 pp. (to appear).
- [Ivr4] V. IVRII. *Sharp Spectral Asymptotics for Operators with Irregular Coefficients. IV. Multidimensional Schrödinger operator with a strong magnetic field. Full-rank case*, 81 pp. (to appear).
- [Ivr5] V. IVRII. *Sharp Spectral Asymptotics for Operators with Irregular Coefficients. IV. Multidimensional Schrödinger operator with a strong magnetic field. Non-Full-rank case*,  $\sim 70$  pp. (in progress).
- [MeRo] M. MELGAARD, G. ROZENBLUM *Eigenvalue asymptotics for weakly perturbed dirac and Schrödinger operators with constant magnetic fields of full rank*, Comm. Partial Differential Equations, **28** (2003) 1&2, 1–52.
- [Rai1] G. RAIKOV. *Border-line eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential Integral Equations*, Operator Theory, **14** (1991) 6, 875–888.
- [Rai2] G. RAIKOV. *Strong electric field eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential*, Lett. Math. Phys., **21** (1991) 1, 41–49.
- [Rai3] G. RAIKOV. *Strong-electric-field eigenvalue asymptotics for the perturbed magnetic Schrödinger operator* Comm. Math. Phys., **155** (1993) 2, 415–428.

- [Rai4] G. RAIKOV. *Semiclassical and weak-magnetic-field eigenvalue asymptotics for the Schrödinger operator with electromagnetic potential*, Ann. Inst. H. Poincaré Phys. Théor., **61** (1994) 2, 163–188.
- [Rai5] G. RAIKOV. *Eigenvalue asymptotics for the Schrödinger operator in strong constant magnetic fields* Comm. Partial Differential Equations, **23** (1998) 9–10, 1583–1619.

Department of Mathematics,  
University of Toronto,  
100 St. George Str.,  
Toronto, Ontario M5S 3G3  
Canada  
ivrii@math.toronto.edu  
Fax: (416)978-4107